ON READABILITY AND MODE READABILITY

Pedro M. G. Ferreira
Departamento de Engenharia Elétrica
PUC-Rio (Reitoria)
22453-900 Rio de Janeiro, Brasil
pedro@arcit.puc-rio.br

Keywords: Readability, Tracking, Loop Stability, Disturbance Rejection, Robustness

Abstract

Readability and Mode Readability are concepts referring to the relationship between the controlled and the measured outputs of a plant. It is shown in this paper that mode readability is not enough if both robust tracking and disturbance rejection are to be achieved. Then, a new concept of readability is proposed, in between the former ones, more adequate to solve the complete servo problem (tracking and disturbance rejection).

1. INTRODUCTION

Very often the controlled output of a plant is different from the measured one and then the question is what should be the relationship between the two in each specific problem.

For the robust multivariable servomechanism problem this question was posed firstly in the mid 70's by Francis and Wonham (1975) and Davison (1976).

Francis and Wonham say that the controlled output \( z \) is readable from the measured output \( y \) if there exist a matrix \( L_R \) with real entries such that

\[
  z = L_R \ y.
\]

(Davison says that in this case \( y \) contains \( z \)).

Those authors assumed readability in order to solve the robust multivariable servomechanism problem, which was done in the state-space domain.

Almost fifteen years later, Sugie and Vidyasagar (1989), addressed the robust multivariable tracking problem under the transfer function point of view. They assumed that the Laplace transforms of \( z \) and \( y \) are related by

\[
  z(s) = L(s) \ y(s),
\]

where the zeros and poles of \( L(s) \) are disjoint from the modes of the signal to be tracked.

(Notice that \( L(s) \) can be unstable and / or improper). They call mode readability this relationship and prove that with this (rather mild) assumption the asymptotic tracking problem has a robust solution. Moreover they go on to conjecture that mode readability is a necessary condition for the solvability of the robust asymptotic tracking problem.

In this paper we will prove in the next section that mode readability is not enough to ensure asymptotic disturbance rejection and in the third section, a new form of readability is proposed, which is between the one of Francis & Wonham and Davison and mode readability, so that asymptotic tracking and disturbance rejection have a robust solution.

Notation and abbreviations: The set of proper and stable rational functions, a principle ideal domain (Vidyasagar, 1985), is denoted by \( S \). The set of matrices with elements in \( X \) is denoted by \( M(X) \). \( R \) is the field of real numbers. Left coprime will be abbreviated by \( l. \ c. \), right coprime will be abbreviated by \( r. \ c. \), "such that" will be \( s.t. \), "with respect to" will be \( w.r.t. \).

2. PRELIMINARIES

Consider the multivariable system depicted in the block diagram at the end of the paper.

In the figure \( z(s) \) and \( r(s) \) are \( q \)-valued vectors, \( u(s) \) is a \( m \)-valued vector and \( y(s) \) is a \( p \)-valued vector. \( r(s) \) is the Laplace transform of the signal to be tracked, \( v(s) \) and \( w(s) \) are Laplace transforms of disturbances to be rejected. \( a \) and \( s \) are the points of failure, actuator and sensor, respectively.

\[
  P(s) = \begin{bmatrix}
    P_1(s) \\
    P_2(s)
  \end{bmatrix}
\]

represents the given plant.

\[
  C(s) = \begin{bmatrix}
    C_1(s) & -C_2(s)
  \end{bmatrix}
\]

is the compensator to be designed.
We will omit henceforth the argument (s) when convenient.

Let \( P_1, P_2, C_1 \) and \( C_2 \) be proper rational matrices and have the appropriate dimensions. \( P_2 \) is assumed strictly proper for convenience in terms of well-posedness and because this is the case in most practical situations. This assumption might be dropped easily. The exogenous signals \( r, v \) and \( w \) are assumed proper. \( P_1 \) and \( P_2 \) are assumed to have full rank. All the factorizations in the paper are over \( M(S) \).

Let \( P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} D^{-1} \), a r.c. factorization.

Let \( C = \begin{bmatrix} C_1 & -C_2 \end{bmatrix} = D_{1c}^{-1} \begin{bmatrix} N_{1c1} & -N_{1c2} \end{bmatrix} \), a l.c. factorization.

We assume that the exogenous signals \( r, v \) and \( w \) have all their poles in the closed right complex plane; those are the relevant poles, since the modes corresponding to stable poles decay asymptotically to zero. This assumption is standard in the literature of the servoloop problem, but might be easily dropped. Define:

\[
r = D_{1c}^{-1} N_{1r} r_0,
\]

where \( D_{1c} \) is a known matrix, \( N_{1r} \) need not be known (as we will see), \( D_{1r} \) and \( N_{1r} \) are l.c. and \( r_0 \) is an arbitrary vector of real numbers.

By the same token, let \( v = D_{1v}^{-1} N_{1v} v_0, w = D_{1w}^{-1} N_{1w} w_0 \), with the same properties as in \( r \).

\( \alpha \) will denote the largest invariant factor of \( D_{1c} \).

We use the standard definition of closed loop stability. It is known that if the closed loop is stable, \( D_{1c}, N_{1c2}, D \) and \( N_2 \) can be chosen, without loss of generality such that

\[
D_{1c} D + N_{1c2} N_2 = I, \tag{1}
\]

where \( I \) is the identity matrix.

Notice from (1) that loop stability implies right coprime-ness of \( N_2 \) and \( D \), and left coprime-ness of \( D_{1c} \) and \( N_{1c2} \). Let us define also \( N_{c2} \) and \( D_c \) r.c. and s.t.

\[
C_2 = N_{c2} D_c^{-1}, \quad D_{1c} \text{ and } N_{1c2} \text{ l.c. such that } P_2 = D_c^{-1} N_{1c2}.
\]

Remark 2.1: Recall (Vidyasagar, 1985) that if \( \|F\|_\infty < 1 \), with \( F \in M(S) \), then \( I + F \) is unimodular.

We have next a technical result:

**Lemma 2.1:** Let \( \Delta_D, \Delta_N \in M(S) \) be such that \( I + D_{1c} \Delta_D + N_{1c2} \Delta_N \) is unimodular. Then there exist \( \varepsilon_1, \varepsilon_2 \in \mathbb{R} \) and \( Q_1, Q_2 \in M(S) \) such that

\[
(I + D_{1c} \Delta_D + N_{1c2} \Delta_N)^{-1} = I - \varepsilon_1 D_{1c} Q_1 - \varepsilon_2 N_{1c2} Q_2 \tag{2}\]

Conversely, let \( \varepsilon_1, \varepsilon_2 \in \mathbb{R} \), and \( Q_1, Q_2 \in M(S) \) be such that \( I - \varepsilon_1 D_{1c} Q_1 - \varepsilon_2 N_{1c2} Q_2 \) is unimodular. Then there exists \( \Delta_D, \Delta_N \in M(S) \) such that (2) holds.

Moreover, \( \Delta_D = \varepsilon_1 Q_1 (I - \varepsilon_1 D_{1c} Q_1 - \varepsilon_2 N_{1c2} Q_2)^{-1} \), \( \Delta_N = \varepsilon_2 Q_2 (I - \varepsilon_1 D_{1c} Q_1 - \varepsilon_2 N_{1c2} Q_2)^{-1} \),  \( \varepsilon_1 Q_1 = \Delta_D (I + D_{1c} \Delta_D + N_{1c2} \Delta_N)^{-1} \), \( \varepsilon_2 Q_2 = \Delta_N (I + D_{1c} \Delta_D + N_{1c2} \Delta_N)^{-1} \).

**Proof:** The proof of this result is quite straightforward: see Ferreira (1999a)

Remark 2.2:

\( Q_1, Q_2 \in M(S) \) may be chosen arbitrarily and yet \( I - \varepsilon_1 D_{1c} Q_1 - \varepsilon_2 N_{1c2} Q_2 \) will be unimodular, provided \( |\varepsilon_1| \) and \( |\varepsilon_2| \) are sufficiently small. Indeed,

\[
\| \varepsilon_1 D_{1c} Q_1 + \varepsilon_2 N_{1c2} Q_2 \|_\infty \leq \| \varepsilon_1 D_{1c} Q_1 \|_\infty + \| \varepsilon_2 N_{1c2} Q_2 \|_\infty,
\]

\[
\| \varepsilon_1 D_{1c} Q_1 \|_\infty \leq |\varepsilon_1| \| D_{1c} \|_\infty \| Q_1 \|_\infty,
\]

\[
\| \varepsilon_2 N_{1c2} Q_2 \|_\infty \leq |\varepsilon_2| \| N_{1c2} \|_\infty \| Q_2 \|_\infty.
\]

3. MODE READABILITY AND DISTURBANCE REJECTION

In their elegant and important paper, Sugie and Vidyasagar (1989) assume that \( N_1 \) and \( N_2 \) are related by

\[
N_1(s) = L(s) N_2(s), \tag{4}
\]

where the zeros and poles of \( L \) are disjoint from those of \( D_{1r} \). Notice that \( L \) can be improper and unstable (but of course \( N_1 \) is proper and stable, by definition). This
relationship between $N_1$ and $N_2$ is a rather mild one. The authors call it “mode readability”, a pretty much weaker condition than “readability”, assumed by Francis & Wonham (1975) and Davison (1976), in which $L$ is constant, i.e., $L \in M(\mathbb{R})$.

Sugi and Vidyasagar show (theorem 3.1) that the asymptotic tracking problem has a robust solution if and only if the nominal closed loop is stable and

\[ (I - L N_2 N_{i2}) D_{iv}^{-1} \in M(\mathbb{S}) \quad (5) \]

\[ \text{ii) The poles of } L D_c \alpha_r^{-1} \text{ are disjoint from the zeros of } \alpha_r. \quad (6) \]

It is shown next that the necessary and sufficient solution of Sugi and Vidyasagar for the robust asymptotic tracking problem does not work for the asymptotic rejection of $v$ if $L D_c$ is unstable.

**Lemma 3.1:** There is no asymptotic rejection of $v$ if $L D_c$ is unstable

**Proof:**

From the block diagram, with $r = 0$, $w = 0$, we have

\[ u = -c_2 (I + P_2 C_2)^{-1} P_2 v = -N c_2 (D_1 D_c + N_{i2} N_{c2})^{-1} N_{i2} v. \]

If the loop is stable, $D_c$ and $N_{c2}$ can be chosen s.t.

\[ D_1 D_c + N_{i2} N_{c2} = I \quad (7) \]

Moreover, the coprime factors of $P_2$ and $C_2$ can always be chosen s.t.

\[ \begin{bmatrix} D_1 - N_{i2} \\ N_{i2} \\ D_{ic} \\ D_{c} \\ -N_{c2} \\ D \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ 0 \\ I \end{bmatrix} \]

(8a)

Commuting the left hand side of (8a), we obtain four additional equalities. Call this

(8b)

In view of this, we have:

\[ z = P_1 (v + u) = N_1 D^{-1} I - N_{c2} N_{i2}) v = N_1 D_{ic} v = L N_2 D_{ic} v = L D_c N_{i2} v = L D_c N_{i2} D_{iv}^{-1} N_{i2} v. \]

Hence asymptotic rejection of $v$ implies

\[ L D_c N_{i2} D_{iv}^{-1} \in M(\mathbb{S}). \quad (9) \]

On the other hand, asymptotic tracking implies (see (5)) that $N_2$ is square or flat. Indeed, if $N_2$ were to have more rows than columns, $L N_2 N_{i2}$ would not have full rank and (5) would be impossible.

Now let $L D_c = B A^{-1}$, a r.c. factorization. Then,

\[ L D_c N_{i2} D_{iv}^{-1} = B A^{-1} N_{i2} D_{iv}^{-1}. \]

But since $N_{i2}$ is square or flat, we may assume that $A$ and $N_{i2}$ are l.c. (If this were not the case, we might perturb $P_2$ slightly, obtaining the coprimeness). Hence (9) is not obtained, proving the lemma.

4. A NEW FORM OF READABILITY

In view of this result, we will assume that $P_1$ and $P_2$ are related by

\[ N_1 = L N_2, \quad \text{with } L \in M(\mathbb{S}). \quad (10) \]

Hence our hypothesis on the relationship between $P_1$ and $P_2$ is more general than the readability of Francis & Wonham and Davison and less general than the mode readability of Sugi and Vidyasagar.

With the above assumption on $L$, it is clear that condition (6) is replaced by

\[ L D_c \alpha_r^{-1} \in M(\mathbb{S}). \quad (6') \]

Clearly, nominal stability of the loop, (5) and (6') are a sufficient condition for robust asymptotic tracking when $L$ is given by (10). The proof of the necessity can be established almost in the same way as in Sugi and Vidyasagar (1989) or using the arguments of Ferreira (1999b).

**Lemma 4.1:**

With $L$ given in (10), the robust asymptotic rejection of $v$ is achieved if $D_{ic} D_{iv}^{-1} \in M(\mathbb{S})$.

**Proof:**

Let $\Delta_v$ and $\Delta_D \in M(\mathbb{S})$ be the perturbations of $N_2$ and $D$, respectively. The admissible $\Delta_N$ and $\Delta_D$ are such that $D_{ic} (D + \Delta_D) + (N_{i2} N_2 + \Delta_N) v = I + D_{ic} D_D + N_{i2} \Delta_N$ is unimodular. It is clear that these perturbations are arbitrary, provided that the norms of $\Delta_v$ and $\Delta_D$ are sufficiently small.

Define $M := I + D_{ic} D_D + N_{i2} \Delta_N$.

\[ u = - (D + \Delta_D)^{-1} N_{i2} (N_2 + \Delta_N) (D + \Delta_D)^{-1} v \]

\[ z = P_1 (v + u) = L (N_2 + \Delta_N) (D + \Delta_D)^{-1} [I - (D + \Delta_D)^{-1} N_{i2} (N_2 + \Delta_N) (D + \Delta_D)^{-1}] v \]

\[ = L (N_2 + \Delta_N) [I - M^{-1} N_{i2} (N_2 + \Delta_N)] (D + \Delta_D)^{-1} v \]

\[ = L (N_2 + \Delta_N) [I - M^{-1} N_{i2} (N_2 + \Delta_N)] (D + \Delta_D)^{-1} v \]

\[ = L (N_2 + \Delta_N) [I - M^{-1} N_{i2} (N_2 + \Delta_N)] (D + \Delta_D)^{-1} v \]
\[ \Delta_D^{-1} v = L (N_2 + \Delta_N) M^{-1} (I + D_{iC} \Delta_D + N_{ie2} \Delta_N - N_{ie2} N_2 - N_{ie2} \Delta_N) (D + \Delta_D)^{-1} v \]

But in view of (1),
\[ z = L (N_2 + \Delta_N) M^{-1} D_{ie} (D + \Delta_D) (D + \Delta_D)^{-1} v. \]

In view of (2), (3b) and (8b):
\[ z = L N_2 \left( I - \varepsilon_1 D_{ie} Q_1 - \varepsilon_2 N_{ie2} Q_2 \right) D_{ie} v + \varepsilon_2 L Q_2 M M^{-1} D_{ie} v \]
\[ = L N_2 D_{ie} v - \varepsilon_1 L N_2 D_{ie} Q_1 D_{ie} v + \varepsilon_2 L (I - N_2 N_{ie2}) Q_2 D_{ie} v \]
\[ = (L D_c N_{i2} D_{iv}^{-1} - \varepsilon_1 L D_c N_{i2} Q_1 D_{ie} D_{iv}^{-1} + \varepsilon_2 L D_c D_1 Q_2 D_{ie} D_{iv}^{-1}) N_{iv} v_0 \]
\[ = (L D_c D_1 Q_2 D_{ie} D_{iv}^{-1} + \varepsilon_2 L D_c D_1 Q_2 D_{ie} D_{iv}^{-1}) N_{iv} v_0 \in M(S), \forall Q_1, Q_2 \in M(S), \]

completing the proof.

**Remark 4.1:** The set of necessary and sufficient conditions for robust asymptotic rejection of \( v \) is given in Ferreira (1999b).

**Remark 4.2:** It is clear from the last statement in the proof of the previous lemma that if \( L \) is unstable but \( D_c \) is such that \( LD_c \) is stable and satisfies the condition of the lemma, then robust asymptotic rejection of \( v \) is achieved as well.

For the rejection of \( w \) we have:

**Lemma 4.2:** With \( L \) given in (10), the robust asymptotic rejection of \( w \) is achieved if
\[ N_{ie2} D_{iw}^{-1} \in M(S). \]

**Proof:**

With \( M \) defined in (11), we have
\[ z = -L(N_2 + \Delta_N) M^{-1} N_{ie2} w \]
\[ = -L N_2 \left( I - \varepsilon_1 D_{ie} Q_1 - \varepsilon_2 N_{ie2} Q_2 \right) N_{ie2} w - L \]
\[ \Delta_N \left( I - \varepsilon_1 D_{ie} Q_1 - \varepsilon_2 N_{ie2} Q_2 \right) N_{ie2} w \]
\[ = -L N_2 N_{ie2} w + \varepsilon_1 L N_2 D_{ie} Q_1 N_{ie2} w + \varepsilon_2 L N_2 N_{ie2} Q_2 N_{ie2} w - \varepsilon_2 L Q_2 N_{ie2} w, \]

in view of (3b).

Hence,
\[ z = -L N_2 N_{ie2} w + \varepsilon_1 L D_c N_{i2} Q_1 N_{ie2} w - \varepsilon_2 L (I - N_2 N_{ie2}) Q_2 N_{ie2} w \]
\[ = -L N_2 N_{ie2} w + \varepsilon_1 L D_c N_{i2} Q_1 N_{ie2} w - \varepsilon_2 L D_c D_1 Q_2 N_{ie2} w, \]

in view of (8b), concluding the proof.

**Remark 4.3:** The set of necessary and sufficient conditions for robust asymptotic rejection of \( w \) is given in Ferreira (1999b).

**Remark 4.4:** Analogously as in Remark 4.2, it is seen from the proof above that if \( L \) is unstable but \( D_c \) is such that \( LD_c \) is stable, then, with the condition of the lemma fulfilled, robust asymptotic rejection of \( w \) is achieved as well.

**References**


The presented proof of Lemma 3.1 is faulty. Indeed, from (9), we have, in view of (8b)
\[ LN_2 D_{lc} D_{lv}^{-1} \in \mathbf{M(S)}. \]
But,
\[ LN_2 = N_1 \in \mathbf{M(S)}. \]
Hence asymptotic rejection of \( \nu \) takes place if \( D_{lc} D_{lv}^{-1} \in \mathbf{M(S)}. \)
However, Lemma 3.1 can be proved using the proof of Lemma 4.1. Indeed, from the last expression of the proof, we have
\[ z = (LN_2 D_{lc} D_{lv}^{-1} - \varepsilon_1 LN_2 D_{lc} Q_1 D_{lc} D_{lv}^{-1} + \varepsilon_2 LD_c D_l Q_2 D_{lc} D_{lv}^{-1}) N_{lv} \nu_v. \]  
(E-1)
With \( \Delta_N = 0 \) and \( \Delta_D = 0 \), we must have \( LN_2 D_{lc} D_{lv}^{-1} \in \mathbf{M(S)}. \)
So, from (E-1) we must have with \( \Delta_D = 0 \)
\[ LD_c D_l Q_2 D_{lc} D_{lv}^{-1} \in \mathbf{M(S)}. \]  
(E-2)
If \( LD_c \not\in \mathbf{M(S)} \), we will have \( LD_c D_l \not\in \mathbf{M(S)} \) also. (If this were not the case, \( D_l \) could be perturbed at the outset, doing it).
Now notice that no unstable pole of \( L \) can possibly be a "blocking zero" of \( D_{lc} \) because then it would be a blocking zero of \( D_c \) also (\( D_{lc} \) is larger than or with the same dimensions as \( D_c \)), and as consequence we would have \( LD_c \in \mathbf{M(S)} \), contrary to the hypothesis.
Next choose \( Q_2 \) as follows: let \( S_D \) be the Smith form of \( D_{lc} \) and let \( U \) and \( V \) be unimodular s.t. \( D_{lc} = US_DV \). Then, from (E-2), we must have
\[ LD_c D_l Q_2 U S_D V D_{lv}^{-1} \in \mathbf{M(S)}. \]  
(E-3)
Let \( s_o \in \mathbb{C}_+ \) be a pole of \( LD_c \) and suppose that it is a zero of \( D_{lc} \). (As seen above, it cannot be a blocking zero of \( D_{lc} \)). Suppose that \( s_o \) is a zero of the last \( k \) invariant factors of \( S_D \). Then we choose \( Q_2 \) s.t. the last \( k \) columns of \( Q_2 U \) are zero and (E-3) is not obtained, proving the lemma.