Robust Tracking: Relation Between the Numerators, Mode Readability, and Inverse Internal Model

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Abstract—In the robust, multivariable, asymptotic tracking problem with two-output plants, it is shown that if the numerator related to the controlled output is fixed, while the other numerator and the denominator are perturbed, then there is no solution to the problem. If, however, the whole part of the plant related to the controlled output is stable and fixed, while the other part is arbitrarily perturbed, the problem has a solution and the compensator that solves the problem incorporates an "inverse internal model" of the signal to be tracked.

Index Terms—Internal model, robustness, stability, tracking.

I. INTRODUCTION

The linear, multivariable, robust tracking problem has been addressed for more than 25 years, having achieved a degree of maturity in the solution of the main issues. In the mid-1970's, the problem was studied, among others, in the state-space/matrix formulation by [1], in the state-space/geometric approach by [5], in the state-space/Laplace transform by [4], and by [3].


Most recently, [6] addressed the issue of the robustness with respect to perturbations of the compensator in the scalar problem with one-output plant, two-degrees-of-freedom compensator. The present note is motivated by a conjecture made by [7], relating the numerators of the two-output plant.

In this note, after the setup of the problem in Section II, we show in Section III that indeed the problem has no solution if the numerators are unrelated, but in Section IV, it is shown that if the part of the plant related to the controlled output is fixed and stable, while the other part is perturbed, then the problem does have a solution: the compensator must incorporate an "inverse internal model" of the exogenous signal.

II. SETTING UP THE PROBLEM

A. Notation and Abbreviations

The set of proper and stable rational functions, a principle ideal domain, [9], is denoted by $S$. The set of matrices with elements in $S$ is denoted by $M(S)$. $R$ is the field of real numbers. Left coprime will be abbreviated by $l.c.$, right coprime will be abbreviated by $r.c.$, "such that" will be abbreviated by $s.t.$ All the left factors will be denoted with an "l" index, e.g., $D_l$.

In Fig. 1, $x(s)$ and $r(s)$ are $q$-valued vectors, $u(s)$ is a $m$-valued vector and $y(s)$ is a $p$-valued vector

$$ P(s) = \begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix} $$

represents the given plant. $C(s) = [C_1(s) - C_2(s)]$ is the compensator to be designed.

Omitting henceforth the argument ($s$) when convenient, we have:

$$ z = P_1 u, \quad y = P_2 u, \quad u = C_r - C_2 y. $$

$P_1, P_2, C_1,$ and $C_2$ are proper rational matrices and have the appropriate dimensions. $P_2$ is assumed strictly proper for convenience in terms of well-posedness and because this is the case in most practical situations. This assumption might be dropped easily. The exogenous signal $r$ is assumed proper. $P_1$ and $P_2$ are assumed to have full rank.

All the factorizations in the paper are over $S$.

Let

$$ P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} D^{-1} $$

a r.c. factorization.

Let $C = [C_1 - C_2] = D^{-1} \begin{bmatrix} N_{r1} \\ -N_{r2} \end{bmatrix}$, a l.c. factorization.

We assume that the exogenous signal $r$ has all its poles in the closed right complex plane; those are the relevant poles, since the modes corresponding to stable poles decay asymptotically to zero. This assumption is standard in the literature of the servo problem, but might be easily dropped.

$$ r = D_{r1}^{-1} N_{r1} r_0 $$

where

$$ D_{r1} \quad \text{is a known matrix.} $$

$$ N_{r1} \quad \text{need not be known,} $$

$$ D_{r1} \text{ and } N_{r1} \text{ are l.c., and} $$

$$ r_0 \quad \text{is an arbitrary vector of real numbers.} $$

$n$ will denote the largest invariant factor of $D_{r1}$. We use the standard definition of closed loop stability. It is known, [2], that if the closed loop is stable, $D_{r1}, N_{r1}, D,$ and $N_2$ can be chosen, without loss of generality such that

$$ D_{l} D = N_{l1/2}/D_1 = l $$

(1)

here $l$ is the identity matrix.

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Moreover, there exist $N_{c1}$, $D_c$, $D_t$, and $N_{c2}$, corresponding to a r.c. of $C_s$ and l.c. factorization of $P_2$, respectively, s.t.

$$
\begin{bmatrix}
D_t & -N_{c2} \\
N_{c1} & D_c
\end{bmatrix}
\begin{bmatrix}
D_c & N_{c2} \\
N_{c1} & D_c
\end{bmatrix} = \begin{bmatrix} I & 0 \\
0 & I \end{bmatrix}.
\tag{1a}
$$

Commutate the left-hand side (LHS) of (1a) and call it (1b).

Let $H$ be the transfer function matrix between $x$ and $r$. Asymptotic tracking is said to take place if and only if the loop is stable and

$$(I - H)r \in M(S).$$

Straightforward calculations give, in view of (1):

$$H = N_1 N_{c1}.$$  

Perturb the plant, $P \rightarrow P^*$. Let $H^*$ be the resulting transfer matrix between $x$ and $r$. We say that $C$ is a robust tracking compensator if and only if the perturbed closed loop is stable and

$$(I - H^*)r \in M(S)$$  \tag{2*}

whatever be the perturbation in a given set.

Remark 1: Recall, [9], that if $\|F\|_{\infty} < 1$, with $F \in M(S)$, then $I + F$ is unimodular. We have next a technical result:

Lemma: Let $\Delta_D$, $\Delta_N \in M(S)$ be such that $I + D_t \Delta_D + N_{c2} \Delta_N$ is unimodular. Then there exist $\epsilon_1$, $\epsilon_2 \in R$ and $Q_1$, $Q_2 \in M(S)$ such that

$$I + D_t \epsilon_1 \Delta_D + N_{c2} \epsilon_2 \Delta_N \sim I - \epsilon_1 D_t \epsilon_1 Q_1 - \epsilon_2 N_{c2} \epsilon_2 Q_2.$$  \tag{4}

Conversely, let $\epsilon_1$, $\epsilon_2 \in R$ and $Q_1$, $Q_2 \in M(S)$ be such that $I - \epsilon_1 D_t \epsilon_1 Q_1 - \epsilon_2 N_{c2} \epsilon_2 Q_2$ is unimodular. Then there exists $\Delta_D$, $\Delta_N \in M(S)$ such that (4) holds. Moreover,

$$\Delta_D = \epsilon_1 Q_1 (I - \epsilon_1 D_t \epsilon_1 Q_1 - \epsilon_2 N_{c2} \epsilon_2 Q_2)^{-1}$$  \tag{5a}

$$\Delta_N = \epsilon_2 Q_2 (I - \epsilon_1 D_t \epsilon_1 Q_1 - \epsilon_2 N_{c2} \epsilon_2 Q_2)^{-1}$$  \tag{5b}

$$\epsilon_1 Q_1 = \Delta_D (I + D_t \epsilon_1 \Delta_D + N_{c2} \epsilon_2 \Delta_N)^{-1}$$  \tag{5c}

$$\epsilon_2 Q_2 = \Delta_N (I + D_t \epsilon_1 \Delta_D + N_{c2} \epsilon_2 \Delta_N)^{-1}.$$  \tag{5d}

Proof: From (5a) and (5b), we have:

$$\begin{align*}
I + D_t \epsilon_1 \Delta_D + N_{c2} \epsilon_2 \Delta_N \\
= I + D_t \epsilon_1 Q_1 (I - \epsilon_1 D_t \epsilon_1 Q_1 - \epsilon_2 N_{c2} \epsilon_2 Q_2)^{-1} \\
+ N_{c2} \epsilon_2 Q_2 (I - \epsilon_1 D_t \epsilon_1 Q_1 - \epsilon_2 N_{c2} \epsilon_2 Q_2)^{-1} \\
= (I - \epsilon_1 D_t Q_1 - \epsilon_2 N_{c2} Q_2) + (I - \epsilon_1 D_t Q_1 - \epsilon_2 N_{c2} Q_2) \\
= (I - \epsilon_1 D_t Q_1 - \epsilon_2 N_{c2} Q_2)^{-1}
\end{align*}

which is (4).

By the same token, (4) is obtained from (5c) and (5d) also.

In their important paper [7], Sugie and Vidyasagar assume that $N_1$ and $N_2$ are related by

$$N_1(s) = L(s) N_2(s)$$  \tag{6}

where the zeros and poles of $L$ are disjoint from those of $D_t$. Notice that $L$ can be improper and unstable (but of course $N_1$ is proper and stable, by definition). This relationship between $N_1$ and $N_2$ is a rather mild one. The authors call it "mode readibility," a weaker condition than "readability," assumed by [1] and [5], in which $L$ is constant. Sugie and Vidyasagar, allow perturbations of $L$ even though not arbitrary. We omit here the class of allowed perturbations of $L$ for the sake of brevity, remitting it to that paper. They make the following conjecture: the relationship (6) is necessary for robust tracking. Then they prove that the compensator, which solve the problem must be such that $L D_{\alpha_0} \gamma^{-1}$ has its poles disjoint from those of $r$. So, it is generalized the idea that the compensator (with $L$) must incorporate a replicated—-in the multivariable case—internal model of the signal to be tracked.

We show in Section III that if $N_1$ is fixed, while $N_2$ and $D$ are perturbed "arbitrarily," the problem has no solution. In Section IV we show that if $P_1$ is fixed and $P_2$ is "arbitrarily" perturbed, the problem does have a solution only if $P_1$ is stable. In this case we will see that the compensator (with $P_1$) must incorporate an inverse internal model of the signal to be tracked.

III. PERTURBING $N_2$ AND $D$ AND FIXING $N_1$

The allowed perturbations in this note are those in the sense of the previous Lemma, i.e., they are arbitrary, but sufficiently small s.t. $I + D_t \Delta_D + N_{c2} \Delta_N$ is unimodular.

Theorem 1: Perturb $D$ and $N_2$ "arbitrarily" (in the sense defined above), while maintaining $N_1$ fixed. Then the robust tracking problem can be solved.

Proof: Perturb $D \rightarrow D + \Delta_D$ and fix $N_1$ and $N_2$. Then it is easy to obtain $s = (I + D_t \Delta_D)^{-1} N_{c1} r = N_{c1} (I - \epsilon_1 D_t Q_1) N_{c1} r$, in view of (4). Hence,

$$\epsilon = r - s = (I - N_{c1} (I - \epsilon_1 D_t Q_1) N_{c1} r)$$  \tag{7}

$$= (I - N_{c1} N_{c1} r) + N_{c1} \epsilon_1 D_t Q_1 N_{c1} r.$$  \tag{8}

Now, in view of (2), (2*), and (3), asymptotic tracking implies

$$N_{c1} D_t Q_1 N_{c1} r \in M(S).$$  \tag{9}

And from the definition of $r$ we get

$$N_{c1} D_t Q_1 N_{c1} D_t^{-1} \in M(S).$$  \tag{10}

Now, in view of (2) and (3) it is clear that $N_{c1}$ and $D_t$ are r.c.

Let $N_{c1} D_t = A^{-1} B$, a l.c. factorization. It is clear that $A$ and $D_t$ have the same invariant factors. Then, in view of (7), we have

$$N_{c1} D_t Q_1 A^{-1} \in M(S).$$  \tag{11}

Let $S_{\lambda}$ be the Smith form of $A$ and let $U$ and $V$ be unimodular matrices such that $A = U S_{\lambda} V$. Define $Q_{11} = Q_1 V^{-1}$. Then from (8)

$$N_{c1} D_t Q_{11} S_{\lambda}^{-1} \in M(S).$$  \tag{12}

Let $\lambda_1$ be the invariant factors of $A$, $j \in m$, where $m = \{1, 2, \cdots, m\}$. Let $n_1, j \in p$, be the columns of $N_{c1} D_t$. Let $q_{11}$ be the elements of $Q_{11}$. Choose $Q_{11}$ such that

$$q_{11} = 1.$$  \tag{13}

which is a necessary condition for good tracking.
Then, straightforward calculations from (9) give \( n_j a_{m}^{-1} \in M(S), \forall j \in p, or, 
\)
\[
N_1 D_{ot} a_{m}^{-1} \in M(S). \tag{10}
\]

Now, perturb \( N_2 \rightarrow N_2 + \Delta N \), fixing \( D \) and \( N_1 \). From the block diagram we obtain
\[
z = N_1(I + N_1 \varepsilon_2 \Delta N)^{-1} N_1 r = N_1(I - \varepsilon_2 N_1 \varepsilon_2 Q_2) N_1 r
\]
in view of (4). Hence,
\[
e = (I - N_1 \varepsilon_2 N_1 r) + N_1 \varepsilon_2 N_1 \varepsilon_2 Q_2 N_1 r.
\]

So, robust tracking implies, in view of (2), (2*), and (3),
\[
N_1 N_1 Q_2 \varepsilon_2 N_1 r^{-1} \in M(S) \quad \forall Q_2 \in M(S).
\]

Defining matrices \( A, S_A \) as above [after (8)] and choosing an appropriate matrix in the same way as \( Q_1 \), we obtain
\[
N_1 N_1 Q_2 N_1^{-1} a_{m}^{-1} \in M(S). \tag{11}
\]

From (10) and (11), we have
\[
a_{m}^{-1} N_1 [D_{ot}, N_{1c}] \in M(S).
\]

But from (1), \( D_{ot} \) and \( N_{1c} \) are I.C., hence, the last implies \( a_{m}^{-1} N_1 \in M(S) \). Hence there exists \( N_{11} \in M(S) \) such that \( N_1 = a_{m} N_{11} \). But from (2) and (3), there should exist \( W \in M(S) \) such that
\[
N_{11} N_{11} N_{11} a_{m} + W D_{ot} = I.
\]

It is clear that there is no solution for this equation in \( N_{11} \) and \( W \), since \( a_{m}, I \) and \( D_{ot} \) are not I.C., proving the theorem.

Remark 2: A reviewer of a previous version of this note remarks on the result above: “If \( P_1 \) perturbs \( y \) and \( z \) have no relation, then the robust tracking has no solution. This result seems to be obvious because we have no way to obtain any information on \( z \) in this case.”

Example 1: We consider now the following simple scalar example, regarding Theorem 1 and the previous remark \( P_1 = P_2 = 1/(s+1) \), \( r = A/s \), \( A \in R \), \( \Delta D = \delta_1 (s+1)^{-1}, \Delta N = \delta_2 (s+1)^{-1}, \delta_1, \delta_2 \in R \), arbitrary and sufficiently small.

We redraw the block diagram (Fig. 2) in a more appropriate way for the problem at hand. Let \( T_1 \) denote the transfer matrix between the input to \( N_1 \) and the output of \( N_{1c} \). In the nominal situation (no perturbation of \( N_2 \) and \( D \)), we have \( T_1 = 1 \). Let the perturbed signals and parameters (as a result of perturbation of \( D \) and \( N_2 \)) be denoted with a (*), e.g., \( T_1, e^{*} \), etc. It is clear that \( T_1^{*} = 1/(1 + \delta_1 (s+1)^{-1} D, + \delta_2 (s+1)^{-1} N_1 r) \).

\[
\Delta T_1 := T_1^{*} - T_1
\]
\[
= -(\delta_1 (s+1)^{-1} D_1 + \delta_2 (s+1)^{-1} N_2 r)/(1 + \delta_1 (s+1)^{-1} D_1 + \delta_2 (s+1)^{-1} N_2 r).
\]

Now, \( e^{*} = (1 - N_1 N_{1c}) r - N_1 \Delta T_1 N_{1c} r \). So, the contribution to the error due to the perturbation in \( D \) and \( N_2 \) is
\[
\Delta e := e^{*} - e
\]
\[
= -N_1 \Delta T_1 N_{1c} r
\]
\[
= -(s+1)^{-1} N_{1c} (\delta_1 D_1 + \delta_2 N_2) A/(s+1)^{-1} (\delta_1 D_1 + \delta_2 N_2).
\]

In view of the fact that \( s(s+1)^{-1} \) and \( N_{1c} \) are coprime, while \( \delta_1 \) and \( \delta_2 \) are arbitrary, it is clear that \( \Delta e \notin S \), because \( D_1 \) and \( N_{1c} \) are coprime. Therefore, the problem has no solution.

IV. SOLUTION OF THE PROBLEM WITH \( P_1 \) FIXED AND \( P_2 \) PERTURBED

Theorem 2: Perturb \( N_2 \) and \( D \) “arbitrarily” (in the sense above) and fix \( P_1 \). Then the asymptotic tracking problem has a solution only if \( P_1 \) is stable. If this is the case, \( C \) is a robust tracking compensator if and only if it stabilizes the loop and
\begin{align*}
& a) \quad (I - P_1 D N_{1c}) D_{ot}^{-1} \in M(S); \\
& b) \quad P_1 N_{1c} a_{m}^{-1} \in M(S).
\end{align*}

Proof: We prove first that \( P_1 \) has to be stable. Condition a) of the theorem is necessary for the nominal plant. Perturb \( D \rightarrow D + \Delta D \). Then,
\[
e = [I - P_1 (D + \Delta D)(I + D_{ot} \Delta N)^{-1} N_{1c}] r
\]
\[
= [I - (P_1 D + P_1 \Delta D)(I - \varepsilon_1 D_1 Q_1) N_{1c}] r
\]
\[
= [I - P_1 D N_{1c} - (\varepsilon_1 P_1 D D_{ot} Q_1 + P_1 \Delta D (I - \varepsilon_1 D_{ot} Q_1)) N_{1c}] r
\]
\[
= [I - P_1 D N_{1c} + (\varepsilon_1 P_1 D D_{ot} Q_1 - \varepsilon_1 P_1 D_1 Q_1) N_{1c}] r \cdot D_{ot}^{-1} N_{1c} r_0.
\]

So, asymptotic tracking implies in view of a) of the theorem
\[
P_1 (I - D D_{ot} Q_1 N_{1c} D_{ot}^{-1}) \in M(S), \quad \forall Q_1 \in M(S).
\]

And in view of (1b), we have
\[
P_1 N_{1c} N_{1c} D_{ot}^{-1} \in M(S), \quad \forall Q_1 \in M(S).
\]

Choosing appropriate \( Q_1 \)'s as in the proof of Theorem 1, the last implies
\[
P_1 N_{1c} N_{1c} a_{m}^{-1} \in M(S). \tag{12}
\]

Let \( N_{1c} D_{ot}^{-1} \) be an a.c. factorization of \( P_1 \). It is clear that stability of the loop implies the left coprimeness of \( D_1 \) and \( N_{1c} \), because \( D_1 \) is a left divisor of \( D \). Then perturb \( N_{1c} \) at the outset, if necessary, so that (12) is not satisfied. Hence, \( P_1 \) has to be stable.
Fig. 3.

We proceed now to the proof of the necessity of condition b) of the theorem, recalling that condition a) is necessary for the nominal loop. Perturb \( N_2 \). Then,

\[
z = P_1 D(I + N_{l2} \Delta N)^{-1} N_{l1} r = P_1 D(I - \varepsilon_1 N_{l2} Q_2) N_{l1} r.
\]

Hence, asymptotic tracking implies

\[
[I - P_1 D(I - \varepsilon_1 N_{l2} Q_2) N_{l1}] D_{l1}^{-1} \in M(S), \quad \forall Q_2 \in M(S).
\]

And this implies in view of a):

\[
P_1 D N_{l2} Q_2 N_{l1} D_{l1}^{-1} \in M(S), \quad \forall Q_2 \in M(S).
\]

In view of (1b), we have

\[
P_1 N_{l2} D_{l1} Q_2 N_{l1} D_{l1}^{-1} \in M(S), \quad \forall Q_2 \in M(S).
\]

Choosing appropriate \( Q_2 \)'s, it can be shown, as in the proof of Theorem 1, that the last implies

\[
P_1 N_{l2} D_{l1} \alpha_m^{-1} \in M(S). \tag{13}
\]

From this and (12),

\[
P_1 N_{l2}[D_{l1}, N_{l2}] \alpha_m^{-1} \in M(S).
\]

And in view of the left coprimeness of \( D_l \) and \( N_{l2} \), we have

\[
P_1 N_{l2} \alpha_m^{-1} \in M(S). \tag{14}
\]

(Sufficiency): Perturbing \( D \) and \( N_2 \), we have

\[
z = P_1 (D + \Delta D) (I + D_{l1} \Delta D + N_{l2} \Delta N)^{-1} N_{l1} r
\]

\[
e = [I - P_1 (D + \Delta D) (I - \varepsilon_1 D_{l1} Q_1 - \varepsilon_2 N_{l2} Q_2) N_{l1}] r
\]

\[
= (I - P_1 D N_{l1}) r - [P_1 D (-\varepsilon_1 D_{l1} Q_1 - \varepsilon_2 N_{l2} Q_2) + P_1 \Delta D (I - \varepsilon_1 D_{l1} Q_1 - \varepsilon_2 N_{l2} Q_2)] N_{l1} r.
\]

Call \( X \) the first term of the right-hand side (RHS), which according to a), belongs to \( M(S) \). Then, in view of (5a),

\[
e = X - \{P_1 (\varepsilon_1 D_{l1} Q_1 - \varepsilon_2 N_{l2} Q_2) + P_1 \varepsilon_1 Q_1 \}[N_{l1} r]
\]

\[
= X - P_1 \varepsilon_1 (I - D_{l1} r) Q_1 N_{l1} r + P_1 \varepsilon_2 D N_{l2} Q_2 N_{l1} r.
\]

So, in view of (1b), we have

\[
e = X - \varepsilon_1 P_1 N_{l2} N_{l2} Q_2 N_{l1} r + \varepsilon_2 P_1 N_{l2} D_{l1} Q_2 N_{l1} r.
\]

The proof is complete in view of the fact that \( \alpha_m r \in M(S) \). \( \square \)

Remark 3: The meaning of this result was given by the reviewer of the previous remark: "If \( P_1 \) is fixed, the information on \( e \) can be obtained via \( u \) when \( P_2 \) perturbs arbitrarily. This is the main reason why the track can be achieved when \( y \) contains no information on \( e \). In this case, the feedforward control plays the essential role in tracking. Therefore the feedforward control should be designed in such a way that \( y \) does not affect \( u \) with respect to the modes of \( r \) (i.e., \( C_2 \) must contain the blocking zeros with respect to the poles of \( F \))."

Example 2: Let \( P_1 = (s + 1)^{-1} \), \( P_2, r_1, \Delta D, \) and \( \Delta N \) as in the previous example. We redraw the block diagram (Fig. 3).

Let \( E \) be the transfer matrix between the input to \( P_1 \) and the output of \( N_{l2} \). It is easy to see that \( E = T_2 = D \). And in view of (1a), we have

\[
\Delta Y_2 := T_2^* - T_2
\]

\[
= (N_{l2} N_{l2} A - D_{l2} N_{l2} \Delta N)/(1 + D_r A + D_{l2} \Delta N)
\]

\[
= (s + 1)^{-1} N_{l2} (A - (A - 1))/\left(1 + (s + 1)^{-1} \right) D_r (s + 1)^{-1} N_{l2}.
\]

Hence, the contribution to the error, due to the perturbations in the plant is

\[
\Delta e = (s + 1)^{-1} (A - \delta_2 (s + 1)) N_{l2} A/\left(1 + (s + 1)^{-1} \right) D_r (s + 1)^{-1} N_{l2}.
\]

It is clear that the problem is solved if \( s(s + 1)^{-1} A \) is a factor of \( N_{l2} \), as pointed out by Remark 3.

The solvability condition for the problem handled in this section is given next.

Theorem 3: Assume \( P_1 \) fixed and stable. Then there exists a two-degrees-of-freedom compensator, which solves the robust asymptotic tracking problem if, and only if, \( P_1 D \) and \( \alpha_m I \) are left coprime.

Proof: (Only if): From (1b), we have

\[
P_1 D D_{l1} + P_1 N_{l2} N_{l2} = P_1.
\]

But \( P_1 \) has full row rank, otherwise asymptotic tracking would be impossible. Let \( P_1^R \) be a right inverse of \( P_1 \). Then,

\[
P_1 D D_{l1} P_1^R + P_1 N_{l2} N_{l2} P_1^R = I.
\]

Let \( e \) be a mode of \( r \). Then from Theorem 2, b), it is clear in view of the above identity that \( P_1 D(e) \) has full row rank. This is equivalent to saying that \( P_1 D \) and \( \alpha_m I \) are left coprime.

(If): If \( P_1 D \) and \( \alpha_m I \) are left coprime, then \( \det(D(e)) \neq 0 \) for every \( \varepsilon \) s.t. \( \alpha_m(e) = 0 \). So, we may choose \( N_{l2} = \alpha_m X \), satisfying condition b) of Theorem 2, with \( X \) and \( D \), s.t. the loop is stable. On the other hand, the left coprimeness of \( P_1 D \) and \( \alpha_m I \) implies the existence of \( Y \) and \( W \in M(S) \), s.t.

\[
P_1 D Y = W \alpha_m I = I.
\]

But \( \alpha_m I = V D_{l1} \), for some \( V \in M(S) \). Hence, we have

\[
P_1 D Y = W D_{l1} = I.
\]

Condition a) of Theorem 2 is satisfied choosing \( N_{l1} = Y \), completing the proof. \( \square \)
V. CONCLUDING REMARKS

i) The problem handled in Section IV has no solution with one-degree-of-freedom feedback compensator. Indeed, conditions a) and b) of Theorem 1 contradict each other if \( N_{11} = N_{12} \).

ii) The result presented in the last section contrasts evidently with the so-called internal model principle: see [5] and [1] and, more recently, [7]. According to the internal model principle, in order to obtain robust tracking, the compensator must incorporate a replicated—in the multivariable problem—internal model of the exogenous signal.

Now, in condition b) of our Theorem 2, we have an "inverse internal model" in the sense that the exogenous poles affect the numerator of the feedback channel of the compensator, not the denominator of it. See Remark 3 above, explaining the apparent contradiction. Indeed, in our note we assume that \( P_1 \) is fixed, while in the three papers mentioned above, a relationship is assumed between \( P_1 \) and \( P_2 \), namely, \( P_1 = LP_2 \). In the first two papers quoted above, \( L \) is a fixed matrix of real numbers and then it is said that \( \varepsilon \) is "readable" from \( y \). In the paper by Sugie and Vidyasagar [7], \( L \) is a rational matrix, not necessarily proper or stable, and whose zeros and poles are disjoin from the exogenous modes: besides, \( L \) is perturbable in a restricted sense and the authors call \( \varepsilon \) "mode readable" from \( y \). It is clear that mode readability is a weaker condition than readability. Sugie and Vidyasagar believe that mode readability is a necessary condition for robust tracking. We assumed that \( P_1 \) is fixed, while \( P_2 \) is arbitrarily perturbed, so in our assumption there is no mode readability and a fortiori no readability.

iii) A practical situation of Theorem 2 might occur, say, when \( P_1 \) would refer to the digital part, so in most cases virtually unperturbable, of a plant.

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REFERENCES


H∞ State Feedback Control for Discrete Singular Systems

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Abstract—This note deals with the problem of state feedback H∞ control for discrete singular systems. It is not assumed that the singular system under consideration is necessarily regular. The problem we address is the design of a state feedback controller such that the resulting closed-loop system is not only regular, causal, and stable, but also satisfies a prescribed H∞-norm bound condition. In terms of certain matrix inequalities, a necessary and sufficient condition for the solution to this problem is obtained, and a suitable state feedback-control law is also given.

Index Terms—Discrete singular systems, H∞ control, state feedback.

I. INTRODUCTION

The problem of H∞ control for standard state-space systems has received considerable interest over the last decade. This problem is concerned with constructing a controller such that the closed-loop system is stable and the norm of the closed-loop transfer function is minimized.

It has been shown that a solution to this problem for linear time-invariant state-space systems involves solving a set of Riccati equations (see [4], [13], [19], and the references therein). In the context of linear discrete-time state-space systems, the results for continuous systems have been extended, see, e.g., [3], [3], and [17]. Some efforts have also been made to deal with the H∞ controller design for discrete-time state-space systems subjected to plant parameter perturbations [2], [6], [18].

Recently, much attention has been given to the extensions of the results of H∞ control theory for state-space systems to singular systems (also known as descriptor systems, implicit systems, generalized state-space systems, differential-algebraic systems, or semistate systems [1], [9]). For example, Takaba et al. [14] have considered the H∞ control problem for singular systems by using the J-spectral factorization approach ([J, J*]-spectral factorization for discrete singular systems can be found in [8]). Masubuchi et al. [11] have shown that the solution of the H∞ control problem for singular systems can be obtained by solving a set of matrix inequalities. Moreover, Wang et al. [15] have presented the necessary and sufficient conditions based on two generalized algebraic Riccati equations for the solution to the above problem. It should be pointed out that all of the works mentioned above are concerned with the H∞ control problem for continuous singular systems.

Though there are many publications on the H∞ controller design for discrete singular systems, it appears that no effort has been made to extend the available results to the case of discrete singular systems.

In this note, we investigate the problem of H∞ control for discrete singular systems. The singular system under consideration is not assumed to be necessarily regular. This implies that we study the H∞ controller design for the general case of discrete singular systems. The motivation for studying these singular systems can be found in [7], [10], and [12]. The purpose of this note is to design a state feedback controller such that the resulting closed-loop system is regular, causal, and stable while satisfying an H∞ norm condition with a prescribed level. To solve this problem, we first give a necessary and sufficient condition for a discrete singular system to be regular, causal, and stable to